NEWTON–LEIBNIZ FORMULA AND HENSTOCK–KURZWEIL INTEGRAL

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The Newton–Leibniz formula is the cornerstone of calculus. It asserts that

$$\int_a^b f'(t)dt = f|_a^b,$$

where f and f' are defined on $[a,b]$ and $f|_a^b = f(b) - f(a)$. The validity of the formula depends on which integral we are considering. If the integral is Riemann integral or the Lebesgue integral then the formula is valid only if the derivative f' is Riemann or Lebesgue integrable. If the integral is the Henstock–Kurzweil integral, the Denjoy integral or the Perron integral then the formula is valid for every derivative f' (cf. [2] pp. 141, 108, 121).

Our objective is to show that the Henstock–Kurzweil definition of integral is an almost immediate generalization of the Newton–Lebniz formula.

We start with the definition of the derivative. It is a well known and quite trivial fact (cf. [1] pp. 57–8) that the standard definition of the derivative is equivalent to the following "straddle" definition.

Definition 1 (Definition of the derivative)**.** *The derivative of f at point t is $f'(t)$ such that*

$$(\forall\varepsilon)(\exists\delta(t))(\forall u, v \in t \mp \delta(t)) \qquad |f(v) - f(u) - f'(t)(v-u)| < \varepsilon(v-u).$$

Remark 1. *We always presuppose that $\varepsilon > 0$ and $\delta > 0$. Further, $u, v \in t \mp \delta(t)$ means that $u \in \langle t - \delta(t), t\rangle$ and $v \in \langle t, t + \delta(t)\rangle$.*

The main consequence of this definition is the following lemma.

Main Lemma. *If f and f' are defined on $[a,b]$ then*

$$(\forall\varepsilon)(\exists\delta(t))(\forall\dot{\mathcal{P}} < \delta(t)) \qquad \left|S(f', \dot{\mathcal{P}}) - f|_a^b\right| < \varepsilon.$$

Remark 2. *We allways presuppose that*

$$\dot{\mathcal{P}} = \{[x_{i-1}, x_i], t_i\}, \quad 1 \le i \le n$$

is a tagged partition of $[a,b]$ and that $S(f', \dot{\mathcal{P}}) = \sum_{i=1}^n f'(t_i)(x_i - x_{i-1})$. Further, $\dot{\mathcal{P}} < \delta(t)$ means that $\dot{\mathcal{P}}$ is a δ-fine tagged partition.

Proof. From the definition of the derivative f' it follows that for every ε there is a $\delta(t)$ such that for every δ-fine tagged partition $\dot{\mathcal{P}}$ of $[a,b]$

$$|f(x_i) - f(x_{i-1}) - f'(t_i)(x_i - x_{i-1})| < \varepsilon/(b-a).$$

Hence,

$$\left|\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) - f'(t_i)(x_i - x_{i-1})\right| \leq$$
$$\sum_{i=1}^{n} |f(x_i) - f(x_{i-1}) - f'(t_i)(x_i - x_{i-1})| <$$
$$\sum_{i=1}^{n} (\varepsilon/(b-a))(x_i - x_{i-1}) = \varepsilon.$$

If we take into account that

$$\sum_{i=1}^{n} f(x_i) - f(x_{i-1}) = f|_a^b\ , \qquad \sum_{i=1}^{n} f'(t_i)(x_i - x_{i-1}) = S(f', \dot{\mathcal{P}}),$$

it immediately follows that

$$(\forall\varepsilon)(\exists\delta(t))(\forall\dot{\mathcal{P}} < \delta(t)) \quad \left| f|_a^b - S(f', \dot{\mathcal{P}})\right| < \varepsilon,$$

which was to be proved. □

Now, consider all possible functionals $\mathcal{I}$ which are defined on triples (a, b, f') consisting of two real numbers a, b and a derivative f' defined on $[a, b]$. They satisfy the following corollary.

Corollary. *$\mathcal{I}(a, b, f') = f|_a^b$ if and only if*

$$(\forall\varepsilon)(\exists\delta(t))(\forall\dot{\mathcal{P}} < \delta(t)) \quad |S(f', \dot{\mathcal{P}}) - \mathcal{I}(a, b, f')| < \varepsilon.$$

Proof. A trivial consequence of the main theorem. □

Next two definitions emerge naturally from the corollary.

Definition 2 (Definition of the Newton–Leibniz property)**.** *A functional $\mathcal{I}$ has the Newton–Leibniz property if $\mathcal{I}(a, b, f') = f|_a^b$ for every f' defined on $[a, b]$. (We also say that $\mathcal{I}$ satisfies the Newton–Leibniz formula.)*

Definition 3 (Definition of the natural integral for the derivatives)**.** *A functional $\mathcal{I}$ is the natural integral for the derivatives if*

$$(\forall\varepsilon)(\exists\delta(t))(\forall\dot{\mathcal{P}} < \delta(t)) \quad |S(f', \dot{\mathcal{P}}) - \mathcal{I}(a, b, f')| < \varepsilon$$

for every f' defined on $[a, b]$.

Hence, the corollary asserts two things.

(1) If $\mathcal{I}$ is the natural integral for the derivatives then it satisfies the Newton–Lebniz formula. (This is the fundamental theorem for the natural integral.)
(2) If $\mathcal{I}$ satisfies the Newton–Lebniz formula then it is the natural integral for the derivatives. (Our main objective was to prove this consequence of the Newton–Leibniz formula.)

We may conclude.

If we start with the assumption that our integral has to satisfy the Newton–Leibniz formula, then we are forced to the definition of the natural integral.

Why did Riemann miss it 150 years ago? Because he did not start with the assumption that his integral has to satisfy the Newton–Leibniz formula. Quite the contrary, his main interest was in integrating functions which are not derivatives and therefore can not be integrated by applying the Newton–Leibniz formula. Along the way, his integral, as well as the 100 years old Lebesgues' integral, lost the close relationship with the derivative. Namely, there are derivatives which are neither Riemann nor Lebesgue integrable.

What is really peculiar, is the case of Denjoy and Perron. More than 90 years ago they explicitely started with the assumptions that their integrals have to satisfy the Newton–Leibniz formula. Nevertheless, they missed the main lemma and its simple corollary. They were working in the Riemann–Lebesgue tradition and hence did not slip back to the derivatives and their integration. Riemann and Lebesgue lost some derivatives (e.g. $x^2 \sin(1/x^2)$) but they did gain a lot of nonderivatives. Hence, it did not seem a good guess that slipping back to the derivatives would lead to a concept of the integral which applied to the nonderivatives would not only restore the full power of Riemann and Lebesgue, but would also give something even stronger.

And this is exactly what happens when we generalize the natural integral for the derivatives to the integral for arbitrary integrable functions:

Definition 4 (Definition of the Henstock–Kurzweil integral). *A real function f is Henstock–Kurzweil integrable on $[a, b]$ if there is a real number $\mathcal{I}(a, b, f)$ such that*

$$(\forall\varepsilon)(\exists\delta(t))(\forall\dot{\mathcal{P}} < \delta(t)) \quad |S(f, \dot{\mathcal{P}}) - \mathcal{I}(a, b, f)| < \varepsilon.$$

The functional $\mathcal{I}(a, b, f)$ defined on every f integrable on $[a, b]$ is the Henstock–Kurzweil integral.

The Newton–Leibniz formula is an equivalent of the definition of the integral for derivatives. The Henstock–Kurzweil integral is its immediate generalization. Hence, the Henstock–Kurzweil integral is an immediate generalization of the Newton–Leibniz formula.

Today, it is a well known fact (cf. [2] pp. 170–173.) that this straightforward generalization is equivalent to the definitions of Denjoy and Perron. But neither Denjoy nor Perron took this simple road 90 years ago.

References

[1] Bartle, Robert G., "*A modern theory of integration*", Graduate Studies in Math., vol. 32., American Math. Soc., Providence, 2001.

[2] Gordon, Russell A., "*The Integrals of Lebesgue, Denjoy, Perron and Henstock*", Graduate Studies in Math., vol. 4., American Math. Soc., Providence, 1994.